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On the Motion of an Ellipsoid in a Fluid.

BY THOMAS CRAIG, *Fellow of the Johns Hopkins University.*

LET us suppose a solid body of the form of an ellipsoid whose mass is so distributed as to be symmetrical with respect to the three principal planes of the body; that this body be immersed in an infinite mass of frictionless incompressible fluid, the whole system being originally at rest, and that the solid be acted upon by any set of impulsive forces—or by any *impulse*—and the system then left to itself; to determine the resulting motion of the solid body and of the fluid. Concerning the motion of the fluid we can say that it will be subject to a velocity potential, since the entire motion of the fluid being due to that of the solid—or to the motion of a portion of the bounding surface of the fluid—there can be no rotation in any part of the mass.

We will now proceed to the determination of the velocity potential ϕ satisfying the equation

$$\nabla^2\phi = 0,$$

∇^2 as usual standing for the operation

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

As we limit ourselves to a simply connected space, the function ϕ will be single valued.

Let now ξ, η, ζ denote the coordinates of a point with respect to a set of rectangular axes fixed in space, and also let x, y, z denote the coordinates of a point with respect to a similar set of axes fixed in the body, we have then

$$\begin{aligned}\xi &= \alpha + \alpha_1 x + \alpha_2 y + \alpha_3 z,\\ \eta &= \beta + \beta_1 x + \beta_2 y + \beta_3 z,\\ \zeta &= \gamma + \gamma_1 x + \gamma_2 y + \gamma_3 z,\end{aligned}$$

the twelve quantities α, β, γ being functions of the time and position of the body whose geometric meaning is well known.

If we denote by u, v, w the translation velocities of the origin of x, y, z in these respective directions and by p, q, r the components of angular velocity around these same axes, we have for the velocity of the fluid particles relatively to the body

$$
1. \qquad \begin{aligned}
\frac{dx}{dt} &= \frac{\partial\varphi}{\partial x} - u - zq + yr = \mathrm{u}\,, \\
\frac{dy}{dt} &= \frac{\partial\varphi}{\partial y} - v - xr + zp = \mathrm{v}\,, \\
\frac{dz}{dt} &= \frac{\partial\varphi}{\partial z} - w - yp + xq = \mathrm{w}\,.
\end{aligned}
$$

If we denote by (nx), (ny), (nz) the angles which the outer normal to the surface of the body makes with the axes of x, y, z respectively, we have

$$\frac{\partial\varphi}{\partial n} = (u + zq - yr)\cos(nx) + (v + xr - zp)\cos(ny) + (w + yp - xq)\cos(nz)\,.$$

As the fluid is to be at rest at an infinite distance from the body, the first derivatives of ϕ with respect to x, y, z vanish at infinity; and since $\nabla^2\phi = 0$ throughout the space filled by the fluid, and the quantities $\frac{\partial\varphi}{\partial x}$, $\frac{\partial\varphi}{\partial y}$, $\frac{\partial\varphi}{\partial z}$ are continuous throughout the same, we have (Kirchhoff, pg. 225) for ϕ the expression

$$\phi = u\phi_1 + v\phi_2 + w\phi_3 + p\phi_4 + q\phi_5 + r\phi_6\,.$$

The six functions ϕ_1, ϕ_2, . . . ϕ_6 satisfying the equation $\nabla^2\phi = 0$ and also giving at the surface of the body

$$
2. \qquad \begin{aligned}
\frac{\partial\varphi_1}{\partial u} &= \cos(nx)\,, & \frac{\partial\varphi_4}{\partial u} &= y\cos(uz) - z\cos(ny)\,, \\
\frac{\partial\varphi_2}{\partial u} &= \cos(ny)\,, & \frac{\partial\varphi_5}{\partial u} &= z\cos(ux) - x\cos(nz)\,, \\
\frac{\partial\varphi_3}{\partial u} &= \cos(nz)\,, & \frac{\partial\varphi_6}{\partial u} &= x\cos(uy) - y\cos(nx)\,,
\end{aligned}
$$

From these equations we see that the functions ϕ_1, ϕ_2, . . . depend simply on the form of the surface of the body, and not at all on its motion, ϕ being a linear function of the quantities u, v, . . . r.

We shall resume now the examination of the function ϕ, observing that, in addition to the equations already given to be satisfied at the surface of the body and throughout the space filled by the fluid, it must also satisfy for $\lambda_1 = \infty$ the equation

$$
3. \qquad \left[\frac{1}{E^2}\left(\frac{\partial\varphi}{\partial\lambda_1}\right) + \frac{1}{F^2}\left(\frac{\partial^2\varphi^2}{\partial\lambda_2}\right) + \frac{1}{G^2}\left(\frac{\partial\varphi^2}{\partial\lambda_3}\right)\right] = 0.
$$

Denoting by Ω the potential of the ellipsoid upon any point x, y, z, we have for a point at the surface of the body

$$\Omega = \text{const.} - \pi\,(Ax^2 + By^2 + Cz^2)$$

(Kirchhoff, pg. 130), where

4. $$A = abc\int\frac{d\lambda_1}{(a^2+\lambda_1)N}, \quad B = abc\int\frac{d\lambda_1}{(b^2+\lambda_1)N}, \quad C = abc\int\frac{d\lambda_1}{(c^2+\lambda_1)N}$$
$$N = \sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)},$$

and for the derivatives of Ω (ib. pg. 225.)

$$\frac{\partial^2\Omega}{\partial n\partial x} = 2\pi\,(2-A)\cos(nx),$$
5. $$\frac{\partial^2\Omega}{\partial n\partial y} = 2\pi\,(2-B)\cos(ny),$$
$$\frac{\partial^2\Omega}{\partial n\partial z} = 2\pi\,(2-C)\cos(nz).$$

Comparing these expressions with the values previously obtained for $\frac{\partial\varphi_1}{\partial n}$ &c., we have

6. $$\phi_1 = \frac{1}{2\pi(2-A)}\frac{\partial\Omega}{\partial x}, \quad \phi_2 = \frac{1}{2\pi(2-B)}\frac{\partial\Omega}{\partial y}, \quad \phi_3 = \frac{1}{2\pi(2-C)}\frac{\partial\Omega}{\partial z}.$$

All the conditions that ϕ_6 must satisfy with the exception of

$$\frac{\partial\varphi_6}{\partial n} = x\cos(ny) - y\cos(nx)$$

will be satisfied if we write

7. $$\phi_6 = N_3\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right)$$

when N_3 is an arbitrary constant; Kirchhoff shows that this condition will also be satisfied by a proper determination of N_3; the value which he obtains in a very simple manner is

8. $$N_3 = \frac{a^2-b^2}{2\pi\,[2(a^2-b^2)+(A-B)(a^2+b^2)]}, \text{ \&c.}$$

If instead of N_1, N_2, N_3 equal to the values here given we write $2\pi N_1$, &c., it will be a little more convenient, and we now have for ϕ

9. $$\phi = \frac{Au}{A-2}x + \frac{Bv}{B-2}y + \frac{Cw}{C-2}z$$
$$+ N_1(B-C)\,yzp + N_2(C-A)\,zxq + N_3(A-B)\,xyr.$$

The quantities A, B, C are given by the expressions

$$A = -bc\frac{\partial\omega_1}{\partial a},$$
$$B = -ca\frac{\partial\omega_1}{\partial b},$$
$$C = -ab\frac{\partial\omega_1}{\partial c},$$

where
$$\omega_1 = \int \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$
and for convenience of future reference
$$\omega_2 = \int \frac{d\lambda_2}{\sqrt{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}},$$
$$\omega_3 = \int \frac{d\lambda_3}{\sqrt{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)}}.$$

We can arrive at the final form of ϕ in a slightly different manner, which will bring in evidence some interesting properties of the quantities that we are dealing with. For simplicity, suppose that the ellipsoid moves with a constant velocity u in the direction of x, then, as before,
$$\phi = u\phi_1,$$
with the prescribed conditions for ϕ_1, will satisfy all the conditions of the motion, and will give again
$$\phi_1 = \frac{1}{2\pi(2-A)}\,\frac{\partial\Omega}{\partial x}.$$

Write
$$A_1,\ B_1,\ C_1 = \frac{A}{abc},\quad \frac{B}{abc},\quad \frac{C}{abc},$$

Now
$$A + B + C = abc\left\{\int_0^\infty \frac{d\lambda_1}{(a^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right.$$
$$\left.+\int_0^\infty \frac{d\lambda_1}{(b^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}} + \int_0^\infty \frac{d\lambda_1}{(c^2+\lambda_1)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right\}$$
$$= abc\int_0^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\left[\frac{(b^2+\lambda_1)(c^2+\lambda_1)+(a^2+\lambda_1)(c^2+\lambda_1)+(a^2+\lambda_1)(b^2+\lambda_1)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\right]$$
$$= \frac{2\,.\,abc}{abc},$$

or
$$A_1 + B_1 + C_1 = \frac{2}{abc},$$

now

10. $$\phi_1 = \frac{1}{2\pi(2-A)}\,\frac{\partial\Omega}{\partial x} = -\frac{abc}{(2-A)}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}},$$

where χ is the positive root of
$$\frac{x^2}{a^2+\chi} + \frac{y^2}{b^2+\chi} + \frac{z^2}{c^2+\chi} = 1;$$

substituting for A its value, we have again
$$\phi_1 = \frac{\displaystyle\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}}{\displaystyle\int_0^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} - \frac{2}{abc}} = \frac{1}{B_1+C_1}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}$$

and consequently

11. $$\phi = \frac{u}{B_1 + C_1}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}}\,.$$

From this value of ϕ we can readily find for U, V, W the values

12. $$\mathrm{U} = \frac{u}{B_1 + C_1}\left\{\int_\chi^\infty \frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} - \frac{x}{\sqrt{(a^2+\chi_1)^3(b^2+\chi_1)(c^2+\chi_1)}}\,\frac{2x}{a^2+\chi}\times\left[\frac{x^2}{(a^2+\chi)^2}+\frac{y^2}{(b^2+\chi)^2}+\frac{z^2}{(c^2+\chi)^2}\right]^{-1}\right\},$$

$$\mathrm{V} = -\frac{u}{B_1+C_1}\,\frac{x}{\sqrt{(a^2+\chi)^3(b^2+\chi)(c^2+\chi)}}\cdot\frac{2z}{b^2+\chi}\cdot\left[\frac{x^2}{(a^2+\chi)^2}+\frac{y^2}{(b^2+\chi)^2}+\frac{z^2}{(c^2+\chi)^2}\right]^{-1},$$

$$\mathrm{W} = -\frac{u}{B_1+C_1}\,\frac{x}{\sqrt{(a^2+\chi)^3(b^2+\chi)(c^2+\chi)}}\cdot\frac{2y}{c^2+\chi}\cdot\left[\frac{x^2}{(a^2+\chi)^2}+\frac{y^2}{(b^2+\chi)^2}+\frac{z^2}{(c^2+\chi)^2}\right]^{-1},$$

since $$\frac{\partial\chi}{\partial x} = \frac{2x}{a^2+\chi}\left[\frac{x^2}{(a^2+\chi)^2}+\frac{y^2}{(b^2+\chi)^2}+\frac{z^2}{(c^2+\chi)^2}\right]^{-1}, \text{ \&c., \&c.}$$

If we suppose now that

$$\phi = u\phi_1 + v\phi_2 + w\phi_3\,,$$

i. e. that the body move in any direction with the component velocities u, v, w, we see that all we have to do is to determine separately ϕ_2 and ϕ_3 and, substituting in this expression for ϕ, have the motion completely determined. Obviously the functions ϕ_2 and ϕ_3 are given by

$$\phi_2 = \frac{1}{A_1 + C_1}\int_\chi^\infty \frac{y\,d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}}\,,$$

$$\phi_3 = \frac{1}{A_1 + B_1}\int_\chi^\infty \frac{z\,d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}}\,,$$

and thus for a simple motion of translation we have

14. $$\phi = \frac{u}{B_1 + C_1}\int_\chi^\infty \frac{x\,d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} + \frac{v}{C_1 + A_1}\int_\chi^\infty \frac{y\,d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}} + \frac{w}{A_1 + B_1}\int_\chi^\infty \frac{z\,d\alpha_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}}\,.$$

Similarly in the case where there is rotation we find the values

15. $$N_1 = \frac{1}{2\pi}\cdot\frac{1}{\left(\dfrac{2}{abc\,A_2} - (b^2+c^2)\right)}\,, \quad A_2 = \int_0^\infty \frac{d\alpha_1}{(c^2+\lambda_1)\,B_1}\,,$$

$$N_2 = \frac{1}{2\pi}\cdot\frac{1}{\left(\dfrac{2}{abcB_2} - (c^2+a^2)\right)}\,, \quad B_2 = \int_0^\infty \frac{d\alpha_1}{(a^2+\lambda_1)\,C_1}\,,$$

$$N_3 = \frac{1}{2\pi}\cdot\frac{1}{\left(\dfrac{2}{abcC_2} - (a^2+b^2)\right)}\,, \quad C_2 = \int_0^\infty \frac{d\alpha_1}{(b^2+\lambda_1)\,A_1}\,.$$

So that finally we can write for the general value of ϕ the following

16. $$\begin{aligned}\phi = {} & \frac{u}{B_1 + C_1}\int_\chi^\infty \frac{x d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)}} + \frac{v}{A_1 + C_1}\int_\chi^\infty \frac{y d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)}} \\ & + \frac{w}{A_1 + B_1}\int_\chi^\infty \frac{z d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)^3}} \\ & + \frac{(b^2-c^2)\,p}{(b^2+c^2)\,A_2 - \frac{2}{abc}}\int_\chi^\infty \frac{yz d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)^3(c^2+\lambda_1)^3}} \\ & + \frac{(c^2-a^2)\,q}{(c^2+a^2)\,B_2 - \frac{2}{abc}}\int_\chi^\infty \frac{zx d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)(c^2+\lambda_1)^3}} \\ & + \frac{(a^2-b^2)\,r}{(a^2+b^2)\,C_2 - \frac{2}{abc}}\int_\chi^\infty \frac{xy d\lambda_1}{\sqrt{(a^2+\lambda_1)^3(b^2+\lambda_1)^3(c^2+\lambda_1)}}\,.\end{aligned}$$

The approximate values of the quantities A, B, C can be obtained by expanding ω_1 in an infinite series and then performing the indicated differentiations, which is the method employed by Clebsch for determining their values; we can also obtain them directly as elliptic functions capable all of being reduced to depend upon the ϑ-function. If we take χ_1, χ_2, χ_3 as the amplitudes of three elliptic integrals

17. $$\theta_1 = \int \frac{d\chi_1}{\Delta(k, \chi_1)}, \quad \theta_2 = \int \frac{d\chi_2}{\Delta(k, \chi_2)}, \quad \theta_3 = \int \frac{d\chi_3}{\Delta(k, \chi_3)}$$

we shall be able to give A, B, C as elliptic functions of any one of these three quantities θ_1, θ_2, θ_3 and could determine ϕ as a function of all three, *i. e.* supposing the proper relations to exist between these quantities and the parameters λ_1, λ_2, λ_3; these relations must evidently be such as to determine θ_1, θ_2, θ_3 as the so-called Lamé or ellipsoidal coordinates. First to determine A, B and C, write

18. $$\lambda_1 = c^2\frac{h^2 - x^2}{x^2}, \quad \text{when } h^2 = \frac{a^2 - c^2}{c^2}$$

x having all values between zero and infinity. This gives us

19. $$\begin{aligned} a^2 + \lambda_1 &= (a^2 - c^2)\frac{x^2+1}{x^2}, \\ b^2 + \lambda_1 &= \frac{(b^2-c^2)\,x^2 + (a^2-c^2)}{x^2}, \\ c^2 + \lambda_1 &= \frac{a^2 - c^2}{x^2}, \\ d\lambda_1 &= -\frac{2\,(a^2-c^2)}{x^3}\,dx\,. \end{aligned}$$

Assume $\frac{a^2-b^2}{a^2-c^2}=k^2$ and, consequently, for the complementary modulus $\frac{b^2-c^2}{a^2-c^2}=k'^2$; making these substitutions and, for brevity, writing $I=\frac{2abc}{(a^2-c^2)^{\frac{3}{2}}}$, we have

$$\begin{aligned} A &= I\int_0^\infty \frac{x^2dx}{(x^2+1)^{\frac{3}{2}}\sqrt{1+k'^2x^2}}, \\ B &= I\int_0^\infty \frac{x^2dx}{(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}}, \\ C &= I\int_0^\infty \frac{x^2dx}{\sqrt{(x^2+1)(1+k'^2x^2)}}. \end{aligned} \tag*{20.}$$

Again, writing $x=\tan\chi_1$, where χ_1 lies between 0 and $\frac{\pi}{2}$, we have readily

$$\begin{aligned} A &= I\int_0^{\frac{\pi}{2}} \frac{\sin^2\chi_1 d\chi_1}{\Delta(k,\chi_1)} = I\int_0^{\frac{\pi}{2}} \operatorname{sn}^2\theta_1 d\theta_1, \\ B &= \frac{I}{k^2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\Delta^2(k,\chi_1)}-1\right)\frac{d\chi_1}{\Delta(k,\chi_1)} = I\int_0^{\frac{\pi}{2}}\left(\frac{1}{\operatorname{dn}^2\theta_1}-1\right)d\theta_1, \\ C &= I\int_0^{\frac{\pi}{2}} \frac{\sin^2\chi_1}{\cos^2\chi_1}\,\frac{d\chi_1}{\Delta(k,\chi_1)} = I\int_0^{\frac{\pi}{2}} \frac{\operatorname{sn}^2\theta_1}{\operatorname{cn}^2\theta_1}\,d\theta_1. \end{aligned} \tag*{21.}$$

Now we know (Cayley, Elliptic Func., p. 15) that

$$k^2\int \operatorname{sn}^2\theta_1 d\theta_1 = \left(1-\frac{E}{K}\right)\theta_1 - Z\theta_1,$$

and so we have readily for A_1

$$\begin{aligned} A &= \frac{I}{k^2}\left\{\left(1-\frac{E}{K}\right)\theta_1 - Z\theta_1\right\} \\ &= -\frac{I}{k^2}\left\{\frac{k\theta_1}{K}\frac{\partial E}{\partial k} + \frac{\Theta'(\theta_1)}{\Theta(\theta_1)}\right\}, \end{aligned} \tag*{22.}$$

and also by not difficult reductions

$$\begin{aligned} B &= \frac{I}{k^2}\left\{\left(\frac{E}{k'^2K}-1\right)\theta_1 + \frac{1}{k'^2}Z(\theta_1+K)\right\} \\ &= \frac{I}{k^2}\left\{\frac{\theta_1}{k}\,\frac{\partial\log K}{\partial k} + \frac{1}{k'^2}\,\frac{\Theta'(\theta_1+K)}{\Theta(\theta_1+K)}\right\}, \\ C &= \frac{I}{k'^2}\left\{\frac{\partial}{\partial\theta_1}\log H(\theta_1+K) - \frac{E}{K}\theta_1\right\}. \end{aligned} \tag*{22.}$$

The expression for C could be given as depending upon the ϑ-function by the formula

$$H(\theta_1+K) = \varepsilon^{-\frac{\pi}{4K}(K'-2i\theta)}\,\Theta(\theta_1+K),$$

but nothing would be gained thereby. We may just notice the forms which immediately present themselves for the constants A_2, B_2 and C_2, but will not integrate the equations. For brevity, we will here denote $-\frac{2}{(a^2-c^2)^{\frac{5}{2}}}$ by I', and we have by the same transformations employed in reducing the quantities A, B and C,

$$
\begin{aligned}
&A_2 = I'\int\frac{x^4dx}{(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d_1\chi}{\cos^2\chi_1\,\Delta^3(k,\chi_1)}, \\
23.\qquad &B_2 = I'\int\frac{x^4dx}{(x^2+1)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d\chi_1}{\cos^2\chi_1\,\Delta(k,\chi_1)}, \\
&C_2 = I\int\frac{x^4dx}{(x^2+1)(1+k'^2x^2)\sqrt{(x^2+1)(1+k'^2x^2)}} = I'\int\frac{\sin^4\chi_1 d\chi_1}{\Delta^3(k,\chi_1)};
\end{aligned}
$$

and these, by virtue of the equation $\chi_1 = \mathrm{am}\theta_1$ become

$$
24.\quad A_3 = I\int\frac{\mathrm{sn}^2\theta_1}{\mathrm{dn}^2\theta_3}\frac{\mathrm{sn}^2\theta_1}{\mathrm{cn}^2\theta_1}\,d\theta_1, \quad B_2 = I\int\mathrm{sn}^2\theta_1\frac{\mathrm{sn}^2\theta_1}{\mathrm{cn}^2\theta_1}\,d\theta_1, \quad C_2 = I\int\mathrm{sn}^2\theta_1\frac{\mathrm{sn}^2\theta_1}{\mathrm{dn}^2\theta_1}\,d\theta_1.
$$

The values of x, y, and z can, of course, be expressed by means of the functions θ_1, θ_2, θ_3. The values of x^2, y^2 and z^2, expressed in terms of the parameters λ_1, λ_2, λ_3, are (see equations 50. in article on "*the Motion of a Solid in a Fluid;*" this Journal, Vol. II, No. 2.)

$$
\begin{aligned}
&x^2 = \frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)}, \\
25.\qquad &y^2 = \frac{(b^2+\lambda_1)(b^2+\lambda_2)(b^2+\lambda_3)}{(b^2-c^2)(b^2-a^2)}, \\
&z^2 = \frac{(c^2+\lambda_1)(c^2+\lambda_2)(c^2+\lambda_3)}{(a^2-c^2)(b^2-c^2)};
\end{aligned}
$$

performing here operations similar to those that have already been employed in transforming to θ_1, we have, since θ_2 and θ_3 sustain the same kind of a relation to λ_2 and λ_3 that θ_1 does to λ_1,

$$
\begin{aligned}
&x^2 = (a^2-c^2)\frac{\mathrm{dn}^2\theta_2\,\mathrm{sn}^2\theta_3}{\mathrm{sn}^2\theta_1}, &&x = \sqrt{a^2-c^2}\frac{\mathrm{dn}\theta_2\,\mathrm{sn}\theta_3}{\mathrm{sn}\theta_1}; \\
26.\qquad &y^2 = (a^2-c^2)\frac{\mathrm{dn}^2\theta_1\,\mathrm{cn}^2\theta_2\,\mathrm{cn}^2\theta_3}{\mathrm{sn}^2\theta_1}, &&y = \sqrt{a^2-c^2}\frac{\mathrm{dn}\theta_1\,\mathrm{cn}\theta_2\,\mathrm{cn}\theta_3}{\mathrm{sn}\theta_1}; \\
&z^2 = (a^2-c^2)\frac{\mathrm{cn}^2\theta_1\,\mathrm{sn}^2\theta_2\,\mathrm{dn}^2\theta_3}{\mathrm{sn}^2\theta_1}, &&z = \sqrt{a^2-c^2}\frac{\mathrm{cn}\theta_1\,\mathrm{sn}\theta_2\,\mathrm{dn}\theta_3}{\mathrm{sn}\theta_1}.
\end{aligned}
$$

With these values of x, y, z, A, B, C, A_2, B_2, C_2, we could now transform ϕ so that it should be given as a function of elliptic functions, but nothing of interest could come from that operation in the general case where none of the velocities vanish, so we will not attempt it. For the case of $k=0$, or an oblate ellipsoid we have readily

27. $$A=B=\frac{I_0}{2}(\theta_1-\sin\theta_1\cos\theta_1)$$
$$C=I_0(\tan\theta_1-\theta_1)\,.$$

Suppose $k=1$, *i. e.* a prolate ellipsoid, then we have

$$\chi_1=\mathrm{gd}\theta_1$$

and $\Delta(k,\chi_1)=\cos\chi_1=\mathrm{cg}\theta_1$, &c. We have in this case, as is well known,

$$\theta_1=\log\tan\left(\frac{1}{2}\pi+\frac{1}{2}\chi_1\right)=\log\tan\left(\frac{1}{2}\pi+\frac{1}{2}\mathrm{gd}\theta_1\right),$$

and, passing to exponentials and reducing,

$$\varepsilon^{\theta_1}=\frac{\mathrm{cg}\theta_1}{1-\mathrm{sg}\theta_1},$$

and thence

$$\mathrm{sg}\theta_1=\frac{\varepsilon^{\theta_1}-\varepsilon^{-\theta_1}}{\varepsilon^{\theta_1}+\varepsilon^{\theta_1}}=\tanh\theta_1,$$

$$\mathrm{cg}\theta_1=\frac{1}{\varepsilon^{\theta_1}+\varepsilon^{-\theta_1}}\,\mathrm{sech}\,\theta_1\,.$$

Substituting in the general values for A, B and C and we have

28. $$A=I\int\mathrm{sg}^2\theta_1\,d\theta_1=I\int(1-\mathrm{cg}^2\theta_1)\,d\theta_1,$$

or,

$$A=I(\theta_1-\mathrm{sg}\theta_1)=I_1(\theta_1-\tanh\theta_1)\,;$$

and similarly,

28′. $$B=C=I_1\int\frac{\mathrm{sg}^2\theta_1}{\mathrm{cg}^2\theta_1}\,d\theta_1=I_1\int\frac{\tanh^2\theta_1}{\mathrm{sech}^2\theta_1}\,d\theta_1$$
$$=I_1\int\sinh^2\theta_1\,d\theta=\frac{I}{4}\left\{\frac{\sinh^2\theta_1}{2}+(1-2\theta_1)\right\},$$

I_1 being what I becomes for $b=c$.

The motion of the fluid particles relatively to the body is given in rectilinear rectangular coordinates by equations 1., but, as for the problem under consideration it is more convenient to use elliptic coordinates, we will consider equations 39. of the article on "*the Motion of a Solid in a Fluid*" above cited.

We will first assume the case where the ellipsoid has merely a motion of translation in the direction of one of its axes—the axis of x for example; for this case we have

$$v=w=p=q=r=0,$$

and the resulting equations of motion,

29.
$$E^2\frac{\partial\lambda_1}{\partial t}=\frac{\partial\varphi}{\partial\lambda_1}-u\frac{\partial x}{\partial\lambda_1},$$
$$F^2\frac{\partial\lambda_2}{\partial t}=\frac{\partial\varphi}{\partial\lambda_2}-u\frac{\partial x}{\partial\lambda_2},$$
$$G^2\frac{\partial\lambda_3}{\partial t}=\frac{\partial\varphi}{\partial\lambda_3}-u\frac{\partial x}{\partial\lambda_3}.$$

These are the equations given and discussed by Clebsch. We have also for ϕ

$$\phi=\frac{A}{A-2}xu=\frac{Ax}{A-2}\cdot\frac{\partial\alpha}{\partial t},$$

and

$$\frac{\partial x}{\partial\lambda_1}=\frac{1}{2}\frac{x}{a^2+\lambda_1},\quad\frac{\partial x}{\partial\lambda_2}=\frac{1}{2}\frac{x}{a^2+\lambda_2},\quad\frac{\partial x}{\partial\lambda_3}=\frac{1}{2}\frac{x}{a^2+\lambda_3},$$

these can clearly be expressed simply in terms of the elliptic functions—but nothing would be gained by it, as the resulting equations of motion will give rise to integrals of a higher order than elliptic. Making these substitutions in the differential equations of motion and they become after simple reductions

30.
$$\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\cdot\frac{\partial\lambda_1}{\partial\alpha}=\frac{2x}{a^2+\lambda_1}\cdot\frac{2}{A-2}\left\{1-\frac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}\right\},$$
$$\frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}{(a^2+\lambda_2)(b^2+\lambda_2)(c^2+\lambda_2)}\cdot\frac{\partial\lambda_2}{\partial\alpha}=\frac{2x}{a^2+\lambda_2}\cdot\frac{2}{A-2},$$
$$\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(a^2+\lambda_3)(b^2+\lambda_3)(c^2+\lambda_3)}\cdot\frac{\partial\lambda_3}{\partial\alpha}=\frac{2x}{a^2+\lambda_3}\cdot\frac{2}{A-2}.$$

Making

$$\frac{4x}{A-2}\cdot\frac{d\alpha}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}=d\Psi$$

we can write these equations in the simpler form

31.
$$(\lambda_2-\lambda_3)\,d\Psi=\frac{d\lambda_1}{(b^2+\lambda_1)(c^2+\lambda_1)}\cdot\frac{1}{1-\dfrac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}},$$
$$(\lambda_3-\lambda_1)\,d\Psi=\frac{d\lambda_2}{(b^2+\lambda_2)(c^2+\lambda_2)},$$
$$(\lambda_1-\lambda_2)\,d\Psi=\frac{d\lambda_3}{(b^2+\lambda_3)(c^2+\lambda_3)},$$

Multiply these equations respectively by

$$c^2+\lambda_1,\quad c^2+\lambda_2,\quad c_2+\lambda_3,$$

and again by

$$b^2+\lambda_1,\quad b^2+\lambda_2,\quad b^2+\lambda_3,$$

and add each of the sets thus obtained, observing that the quantity

$$\frac{1}{1-\dfrac{2}{(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}}$$

can be written

$$1-\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},$$

and we have for the equations of the motion of the fluid particle relatively to the body

$$\text{32.}\quad \begin{aligned} 2\frac{dy}{y}&=\frac{d\lambda_1}{b^2+\lambda_1}\cdot\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},\\ 2\frac{dz}{z}&=\frac{d\lambda_1}{c^2+\lambda_1}\cdot\frac{1}{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}},\end{aligned}$$

since

$$2\frac{dy}{y}=\frac{d\lambda_1}{b^2+\lambda_1}+\frac{d\lambda_2}{b^2+\lambda_2}+\frac{d\lambda_3}{b^2+\lambda_3},\ \&c.$$

From these we see that the path of the particle lies always in a plane passing through the axis of x; by taking values of λ_1 so large that the fourth powers of the ratios $\sqrt{\frac{a^2}{\lambda_1}}$, $\sqrt{\frac{b^2}{\lambda_1}}$, $\sqrt{\frac{c^2}{\lambda_1}}$, can be neglected, Clebsch shewed that the path of particles very rapidly approached straight lines. By subtracting the second of these equations from the first and integrating we have

$$\text{33.}\quad \log\frac{y}{z}=-(b^2-c^2)\int\frac{d\lambda_1}{(b^2+\lambda_1)(c^2+\lambda_1)\left\{1-\dfrac{1}{2}(A-2)\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}\right\}}.$$

This can be expressed in terms of the elliptic functions, for we have

$$\begin{aligned}&\int\frac{d\lambda_1}{\sqrt{(a^2+\lambda_1)(b^2+\lambda_1)(c^2+\lambda_1)}}=-\frac{1}{2}\,\frac{d\theta_1}{\sqrt{a^2-c^2}},\\ &d\lambda_1=-\frac{1}{2}\,\frac{d\theta_1}{\sqrt{a^2-c^2}}\cdot\frac{\mathrm{sn}^3\theta_1}{\mathrm{cn}\theta_1\mathrm{dn}\theta_1},\\ &(b^2+\lambda_1)(c^2+\lambda_1)=(a^2-c^2)^2\cdot\frac{\mathrm{cn}^2\theta_1\mathrm{dn}^2\theta_1}{\mathrm{sn}^4\theta_1},\\ &A-2=2\left\{\frac{1}{k^2(a^2-c^2)^{\frac{3}{2}}}\left[\left(1-\frac{E}{K}\right)\theta_1-Z\theta_1\right]-1\right\}.\end{aligned}$$

Making these substitutions, we have

$$\text{34.}\quad \log\left(\frac{y}{z}\right)^{\frac{k'^2}{(a^2-c^2)^{\frac{1}{2}}}}=\int\frac{\mathrm{sn}^7\theta_1\,d\theta_1}{\mathrm{cn}^3\theta_1\mathrm{dn}^3\theta_1\left\{1-\left[\dfrac{\left(1-\dfrac{E}{K}\right)\theta_1-Z\theta_1}{k^2(a^2-c^2)^{\frac{3}{2}}}-1\right]\left[(a^2-c^2)^{\frac{3}{2}}\dfrac{\mathrm{cn}\,\theta_1\,\mathrm{dn}\,\theta_1}{\mathrm{sn}^3\theta_1}\right]\right\}}$$

a non-integrable expression. Nothing of interest could be obtained by a further examination of this case, and so we shall leave it, calling attention, however, to the fact that the coordinates of any fluid particle are expressed as functions of the parameter of the confocal ellipsoid upon which it lies. The same is also true in the case of a simple rotation around one axis—as the axis of x; the resulting integral equation of the path of the particle comes in this case also in a form that is too complicated to admit of any discussion. The motion of the particles, very near the surface of the body, would be, in an ordinary liquid, a very important point to determine—but, as in that case, the equations will present still greater difficulties than they do in the case of a perfect fluid, it does not seem as if that problem is susceptible of solution.

We shall now take up the second and more interesting part of the general problem and investigate the motion of the solid body. As there are no external forces acting upon either the body or the fluid, the motion of the system is entirely due to that of the body, and, in consequence, the total energy of the motion is a homogeneous quadratic function of the quantities u, v, w, p, q, r with constant coefficients. On account of the assumed symmetry of figure and distribution of mass of the body, *i. e.* with respect to three mutually perpendicular planes, the expression for the energy of the system will contain only the squares of u, v, &c., and may be written

$$2T = a_{11}u^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2 ,$$

where the constant coefficients a_{ij} depend upon the density of the fluid, the form of the surface of the body, and the moments of inertia about the axes. Write for brevity

$$\frac{\partial T}{\partial u} = U, \ldots . \frac{\partial T}{\partial r} = R .$$

The Kirchhoffian equations for the motion of the body are now

35.
$$\begin{aligned} \frac{\partial U}{\partial t} &= qW - rV, & \frac{\partial P}{\partial t} &= vW - wV + qR - rQ, \\ \frac{\partial V}{\partial t} &= rU - pW, & \frac{\partial Q}{\partial t} &= wU - uW + rP - pR, \\ \frac{\partial W}{\partial t} &= pV - qU, & \frac{\partial R}{\partial t} &= uV - vU + pQ - qP. \end{aligned}$$

From these equations we have the well known integrals

36.
$$\begin{aligned} &2T = \text{const.}, \\ &U^2 + V^2 + W^2 = \text{const.} = L, \\ &UP + VQ + WR = \text{const.} = M. \end{aligned}$$

For the assumed case the equations of motion become

$$37.\qquad \begin{aligned} a_{11}\frac{\partial u}{\partial t} &= a_{33}wq - a_{22}vr, & a_{44}\frac{\partial p}{\partial t} &= (a_{33}-a_{22})\,vw + (a_{66}-a_{55})\,qr, \\ a_{22}\frac{\partial v}{\partial t} &= a_{11}ur - a_{33}wp, & a_{55}\frac{\partial q}{\partial t} &= (a_{11}-a_{33})\,wu + (a_{44}-a_{66})\,pr, \\ a_{33}\frac{\partial w}{\partial t} &= a_{22}vp - a_{11}uq, & a_{66}\frac{\partial r}{\partial t} &= (a_{22}-a_{11})\,uv + (a_{55}-a_{44})\,pq. \end{aligned}$$

Before going on to the integration of these equations for special cases, it will be desirable to determine the values of the constants a_{ij}. These quantities are made up of two parts depending, as they do, upon the energies of the solid and the fluid; that part of a_{11} which depends upon the solid merely is m, or the mass of the solid; and the same is true for a_{22} and a_{33}: the parts of a_{44}, a_{55}, a_{66} depending upon the body are

$$\frac{m}{5}(b^2+c^2), \quad \frac{m}{5}(c^2+a^2), \quad \frac{m}{5}(a^2+b^2).$$

For the parts depending upon the energy of the fluid we have, denoting any one of them by a'_{ij},

$$a'_{ij} = \rho\int \Phi_j \frac{\partial \varphi_i}{\partial u}\, d\sigma$$

(see article above cited, or Kirchhoff's *Math. Phys.*); the integration is of course to be extended over the entire surface of the ellipsoid.

The values of these constants have been already obtained by several writers, and I will merely write them down—they are:

$$38.\qquad \begin{aligned} a'_{11} &= m\frac{\rho}{\rho_0}\frac{A}{B+C}, \\ a'_{22} &= m\frac{\rho}{\rho_0}\frac{B}{C+A}, \qquad \rho_0 \text{ denoting the density of the solid.} \\ a'_{33} &= m\frac{\rho}{\rho_0}\frac{C}{A+B}. \end{aligned}$$

$$39.\qquad \begin{aligned} a'_{44} &= \frac{m}{5}(b^2+c^2)\left\{1+\frac{\rho}{\rho_0}\cdot\frac{B-C}{B-C+\dfrac{b^2-c^2}{b^2+c^2}(A+B+C)}\right\}, \\ a'_{55} &= \frac{m}{5}(c^2+a^2)\left\{1+\frac{\rho}{\rho_0}\cdot\frac{C-A}{C-A+\dfrac{c^2-a^2}{c^2+a^2}(A+B+C)}\right\}, \\ a'_{66} &= \frac{m}{5}(a^2+b^2)\left\{1+\frac{\rho}{\rho_0}\cdot\frac{A-B}{A-B+\dfrac{a^2-b^2}{a^2+b^2}(A+B+C)}\right\}. \end{aligned}$$

The first three of these are the same in form as the values deduced by Mr. Ferrers in a recent number of the Quart. Jour.; the last three may be obtained from his values or from the values already given for N_1, N_2, N_3.

The values of these constants are thus seen to depend upon elliptic functions, and they can be readily determined from the previously given values of A_2, B_2, C_2, or they may be reduced to depend merely upon A, B, C. Consider for a moment the last three, viz: a_{44}, a_{55}, a_{66}; they are the moments of inertia of the ellipsoid, with respect to its axes, increased by certain quantities depending upon the density of the fluid and also upon the density and ellipsoidal shape of the immersed body. Conceive another ellipsoid whose semi-axes are a_1, b_1, c_1 and of mass m; the magnitude of the axes to be determined from the conditions

40.
$$\frac{m}{5}(b_1^2 + c_1^2) = a_{44},$$
$$\frac{m}{5}(c_1^2 + a_1^2) = a_{55},$$
$$\frac{m}{5}(a_1^2 + b_1^2) = a_{66}.$$

Denote for brevity the fractional quantities in the expressions for a_{44}, a_{55}, a_{66} by ζ_1, ζ_2, ζ_3, then we have for the determination of a_1^2, b_1^2, c_1^2 the equations

41.
$$b_1^2 + c_1^2 = (b^2 + c^2)(1 + \zeta_1),$$
$$c_1^2 + a_1^2 = (c^2 + a^2)(1 + \zeta_2),$$
$$a_1^2 + b_1^2 = (a^2 + b^2)(1 + \zeta_3);$$

these give readily

42.
$$a_1^2 = a^2 + \frac{(a^2 + b^2)\zeta_3 + (c^2 + a^2)\zeta_2 - (b^2 + c^2)\zeta_1}{2} \equiv a^2 + \tau_1,$$
$$b_1^2 = b^2 + \frac{(a^2 + b^2)\zeta_3 - (c^2 + a^2)\zeta_2 + (b^2 + c^2)\zeta_1}{2} \equiv b^2 + \tau^2,$$
$$c_1^2 = c^2 + \frac{-(a^2 + b^2)\zeta_3 + (c^2 + a^2)\zeta_2 + (b^2 + c^2)\zeta_1}{2} \equiv c^2 + \tau_3;$$

and we have also for the new density

$$\rho'_0 = \rho_0 \frac{abc}{a_1 b_1 c_1} = \rho_0 \frac{abc}{\sqrt{(a^2 + \tau_1)(b^2 + \tau_2)(c^2 + \tau_3)}}.$$

If the ellipsoid is oblate

$$a_1^2 + b_1^2 = a^2 + b^2,$$

if it is prolate,

$$b_1^2 + c_1^2 = b^2 + c^2.$$

A further study of the properties of this ellipsoid would undoubtedly prove of interest, as it seems to possess, as will be seen shortly, properties somewhat analogous to "Poinsot's ellipsoid" in rigid dynamics.

The transformation to Lagrangian, or generalised coordinates, gives rise to several interesting results, and before going further with the discussion in Eulerian coordinates it will be desirable to make this transformation. I will write down at once the values of u, v, w, p, q, r in terms of the generalised coordinates $\dot{x}$, $\dot{y}$, $\dot{z}$, θ, ϕ, ψ, $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, as they are obtained in Thomson & Tait's *Nat. Phil.*, (new edition), and also in "*The Application of Generalised coordinates to the Kinetics of a Material System*," by Watson and Burbury; they are

$$\begin{aligned} u &= (\cos\theta\cos\phi\cos\psi - \sin\theta\sin\psi)\,\dot{x} \\ &\quad + (\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi)\,\dot{y} - \sin\theta\cos\phi\dot{z}\,, \\ \alpha. \qquad v &= -(\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\,\dot{x} \\ &\quad + (\cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi)\,\dot{y} + \sin\theta\sin\phi\dot{z}\,, \\ w &= \sin\theta\cos\psi\dot{x} + \sin\theta\sin\psi\dot{y} + \cos\theta\dot{z}\,, \end{aligned}$$

and for the rotations the well known values

$$\begin{aligned} p &= \sin\phi\dot{\theta} - \sin\theta\cos\phi\dot{\psi}\,, \\ \beta. \qquad q &= \cos\phi\dot{\theta} + \sin\theta\sin\phi\dot{\psi}\,, \\ r &= \dot{\phi} + \cos\theta\dot{\psi}\,. \end{aligned}$$

The Lagrangian equations of motion are also, no external forces acting,

$$\begin{aligned} &\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{x}} = 0\,, \quad \frac{\partial}{\partial t} - \frac{\partial T}{\partial\dot{y}} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{z}} = 0\,, \\ \gamma. \quad &\frac{\partial}{\partial t}\frac{\partial T}{\partial\dot{\theta}} - \frac{\partial T}{\partial\theta} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\varphi} - \frac{\partial T}{\partial\dot{\varphi}} = 0\,, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\psi}} - \frac{\partial T}{\partial\psi} = 0\,; \end{aligned}$$

these equations can be found in either of the above mentioned books, or in Routh's Rigid Dynamics, and are too well known to need any explanation. The values of $\dot{x}$, $\dot{y}$, $\dot{z}$ could be obtained by direct solutions of equations α, but that would be a long process, and it is obvious from their forms that the desired quantities can be otherwise obtained. Multiply u by $\cos\theta\cos\phi$, v by $\cos\theta\sin\phi$, w by $\sin\theta$; and again multiply u by $\sin\phi$, v by $\cos\phi$; and finally u by $-\sin\theta\cos\phi$, v by $\sin\theta\cos\phi$, and w by $\cos\theta$ and in each case add the results; we have then

$$\begin{aligned} \dot{x}\cos\psi + \dot{y}\sin\psi &= u\cos\theta\cos\phi - v\cos\theta\sin\phi + w\sin\theta \\ -\dot{x}\sin\psi + \dot{y}\cos\psi &= u\sin\phi + v\cos\phi\,, \\ \dot{z} &= -u\sin\theta\cos\phi + v\sin\theta\cos\phi + w\cos\theta \\ &= (v-u)\sin\theta\cos\phi + w\cos\theta\,, \end{aligned}$$

the values of $\dot{x}$, $\dot{y}$ and $\dot{z}$ obtained from these equations are readily found to be

$$\begin{aligned} &\dot{x} = (\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi)\, u \\ &\qquad - (\cos\theta\sin\phi\cos\psi + \cos\phi\sin\psi)\, v + \sin\theta\cos\psi w\,, \\ 44.\quad &\dot{y} = (\cos\theta\cos\phi\sin\psi + \sin\phi\cos\psi)\, u \\ &\qquad - (\cos\theta\sin\phi\sin\psi - \cos\phi\cos\psi)\, v + \sin\theta\sin\psi w\,, \\ &\dot{z} = -u\sin\theta\cos\phi + v\sin\theta\cos\phi + w\cos\theta\,. \end{aligned}$$

Having thus determined $\dot{x}$, $\dot{y}$, $\dot{z}$ as functions of θ, ϕ, ψ, if we can now, from equations β, determine these latter quantities as functions of $\dot{\theta}$, $\dot{\phi}$, $\dot{\psi}$, we shall be in condition to integrate these differential equations and so determine the path described by any point of the body during its motion through the fluid. We know that whatever motion the body may have at any time that we can conceive the motion as due to a properly applied *impulse* at the beginning of the motion. It will simplify the work to assume the axis of the impulse as coinciding with one of the axes of reference, as the axis of z; then calling L, see equations 36, the total momentum of the impulse we will have from equations α—the components of momentum in the directions of x and y being zero,

$$45.\qquad \frac{\partial T}{\partial \dot{x}} = 0\,, \quad \frac{\partial T}{\partial \dot{y}} = 0\,, \quad \frac{\partial T}{\partial \dot{z}} = L, \text{ (a constant)}.$$

We have further, Thomson & Tait, § 221, for the Eulerian components of momentum $a_{11}u$, $a_{22}v$, $a_{33}w$

$$\begin{aligned} a_{11}u\cos\phi - a_{22}v\sin\phi &= -L\sin\theta\,, \\ 46.\qquad a_{11}u\sin\phi + a_{22}v\cos\phi &= 0\,, \\ a_{33}w &= L\cos\theta\,; \end{aligned}$$

from these follow

$$\begin{aligned} a_{11}u &= -L\sin\theta\cos\phi\,, \\ 47.\qquad a_{22}v &= L\sin\theta\sin\phi\,, \\ a_{33}w &= L\cos\theta\,. \end{aligned}$$

From these we have, for the first three terms in the expression for the energy,

$$\begin{aligned} &a_{11}u^2 + a_{22}v^2 + a_{33}w^2 = \\ 48.\qquad &L^2\left[\sin^2\theta\left(\frac{\cos^2\varphi}{a_{11}} + \frac{\sin^2\varphi}{a_{22}}\right) + \frac{\cos^2\theta}{a_{33}}\right] = G\,. \end{aligned}$$

For the last three terms containing the squares of the angular velocities,

$$49.\quad a_{44}\,(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 + a_{55}\,(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 + a_{66}\,(\dot{\psi}\cos\theta + \dot{\phi})^2;$$

this quantity is identical with the expression for the energy of a free rigid body rotating about its center of inertia (Thomson & Tait, p. 314), the moments of inertia being a_{44}, a_{55}, a_{66}. We have for the total kinetic energy,

50. $$2T = G + a_{44}(\dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi)^2 + a_{55}(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi)^2 + a_{66}(\dot{\psi}\cos\theta + \dot{\phi})^2;$$

from this it is clear that the motion of the ellipsoid in the fluid is identical with the motion of another ellipsoid of semi-axes a_1, b_1, c_1 rotating about its center under a potential G, or, the entire kinetic energy of the system of fluid and ellipsoid is equal to the entire energy of an ellipsoid of semi-axes a_1, b_1, c_1 and density ρ'_0 rotating about its center under a potential G.

If the ellipsoid be of revolution around the axis of z the expressions for $\dot{x}$, $\dot{y}$, $\dot{z}$ become much simplified; we have in that case

$$a_{11} = a_{22}, \quad a_{44} = a_{55},$$

and, as we have already seen,

$$a_{66} = \frac{m}{5}(a^2 + b^2).$$

The values of $\dot{x}$, $\dot{y}$, $\dot{z}$ are now

51. $$\begin{aligned} \dot{x} &= -L\sin\theta\cos\theta\cos\psi\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right), \\ \dot{y} &= L\sin\theta = \cos\theta\sin\psi\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right), \\ \dot{z} &= L\sin^2\theta\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right) + \frac{L}{a_{33}}. \end{aligned}$$

[A very interesting problem is solved by Thomson & Tait, viz: to find the motion of a solid of revolution in a fluid which so moves that the axis of revolution shall always be in one plane—the body has a motion of translation in the direction of the axis of x and an angular motion around this axis—and it is found that the resulting equation of motion is the same in form as that of the common pendulum, the mass, moment of inertia and length of the pendulum depending upon the mass and figure of the body and the density of the liquid.]

The substitution of the relations $a_{11} = a_{22}$ and $a_{44} = a_{55}$ in the expression for $2T$ cause this quantity to become

52. $$2T = L^2\sin^2\theta\left(\frac{1}{a_{11}} - \frac{1}{a_{33}}\right) + \frac{L^2}{a_{33}} + a_{44}(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + a_{66}(\dot{\phi} + \cos\theta\dot{\psi})^2$$

containing neither ϕ nor ψ, and, consequently, cause the fifth and sixth of equations γ to become

$$\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\varphi}} = 0, \quad \frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot{\psi}} = 0,$$

or,

$$\frac{\partial T}{\partial\dot{\varphi}} = \text{const.} \quad \frac{\partial T}{\partial\dot{\psi}} = \text{const.};$$

equating then to constants these derivatives of T and we have

53.
$$\dot\phi + \cos\theta\dot\psi = \mu_1\,,$$
$$a_{44}\sin^2\theta\dot\psi + a_{66}\mu_1\cos\theta = \mu_2\,,$$

μ_1 denoting the angular velocity around the axis of z and μ_2 the component angular momentum about the same axis. The equation

$$\frac{\partial}{\partial t}\cdot\frac{\partial T}{\partial\dot\theta}=\frac{\partial T}{\partial\theta}\,,$$

now merely expresses what we already know, that T, the kinetic energy, is constant.

The expression for the energy may now be written

$$L^2\sin^2\theta\left(\frac{1}{a_{11}}-\frac{1}{a_{33}}\right)+\frac{L^2}{a_{33}}+a_{44}(\dot\theta^2+\sin^2\theta\dot\psi^2)+a_{66}\mu_1^2\,.$$

The steady motion of the solid is easily obtained from the original expression for the energy, together with the values already given for u, v, w, p, q, r and for $\dot x$, $\dot y$ and $\dot z$. We must have $\theta=0$ and consequently $\frac{\partial T}{\partial\theta}=0$; since $\frac{\partial T}{\partial\varphi}$ and $\frac{\partial T}{\partial\psi}$ are identically zero, they of course do not come into the question, we must also have $\dot\psi=$ const., say η.

$$\frac{\partial T}{\partial\theta}=\frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial\theta}+\ldots.\frac{\partial T}{\partial r}\,\frac{\partial r}{\partial\theta}\,;$$

$$\frac{\partial T}{\partial u}=a_{11}u\,,\quad\frac{\partial T}{\partial p}=a_{44}p\,,$$
$$\frac{\partial T}{\partial v}=a_{33}v\,,\quad\frac{\partial T}{\partial q}=a_{55}q\,,$$
$$\frac{\partial T}{\partial w}=a_{33}w\,,\quad\frac{\partial T}{\partial r}=a_{66}r\,;$$

since $a_{22}=a_{33}$
and $a_{55}=a_{66}$.

$$\frac{\partial u}{\partial\theta}=-(\sin\theta\cos\phi\cos\psi\dot x+\sin\theta\cos\phi\sin\psi\dot y+\cos\theta\cos\phi\dot z)\,,$$
$$\frac{\partial v}{\partial\theta}=\quad(\sin\theta\sin\phi\cos\dot\psi+\sin\theta\sin\phi\sin\psi\dot y+\cos\theta\sin\phi\dot z),$$
$$\frac{\partial w}{\partial\theta}=\quad(\cos\theta\cos\psi\dot x+\cos\theta\sin\psi\dot y=\sin\theta\dot z),$$
$$\frac{\partial p}{\partial\theta}=-\cos\theta\cos\phi\dot\psi\,,$$
$$\frac{\partial q}{\partial\theta}=\quad\cos\theta\sin\phi\dot\psi\,,$$
$$\frac{\partial r}{\partial\theta}=-\sin\theta\dot\psi\,,$$

Substituting all these values in $\frac{\partial T}{\partial \theta}$, and also giving p and q their values in the generalized coordinates, and this becomes

$$\begin{aligned}\frac{\partial T}{\partial \theta} = &- a_{11}(u \cos \phi - v \sin \phi)(\sin \theta \cos \psi \dot{x} + \sin \psi \dot{y} + \cos \theta \dot{z}) \\ &+ a_{33} w (u \cos \phi - v \sin \phi) \\ &- a_{44} (\sin \phi \dot{\theta} - \sin \theta \cos \phi \dot{\psi}) \cos \theta \cos \phi \dot{\psi} \\ &+ a_{44} (\cos \phi \dot{\theta} \text{ m} \sin \theta \sin \phi \dot{\psi}) \cos \theta \sin \phi \dot{\psi} \\ &+ a_{66} r \sin \theta \dot{\psi},\end{aligned}$$

since

$$\begin{aligned}u \cos \phi - v \sin \phi &= \dot{x} \cos \psi \cos \theta + \dot{y} \sin \psi \cos \theta - \dot{z} \sin \theta \\ &= (a_{33} - a_{11})(u \cos \phi - v \sin \phi)\, w \\ &+ a_{44} \sin^2 \theta \dot{\psi}^2 - a_{66} r \cos \theta \dot{\psi} \\ &= L^2 \sin \theta \cos \theta \left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) + a_{44} \sin \theta \cos \theta \dot{\psi}^2 - a_{66} r \sin \theta \dot{\psi}.\end{aligned}$$

Calling now the integral of $\theta' = 0$, $\theta = \theta'$, then $\cos \theta = \cos \theta'$, we had $\dot{\psi} = \eta$, therefore, $\cos \psi = \cos \eta t$, &c. The equation $\frac{\partial T}{\partial \theta} = 0$, becomes now

54. $$a_{44} \cos \theta' \eta^2 - a_{66} r \eta + L^2 \left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) \cos \theta' = 0.$$

which gives

55. $$r = \frac{L^2 \left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) \cos \theta'}{a_{66} \eta} + \frac{a_{44}}{a_{66}} \cos \theta \eta,$$

the value of the angular velocity necessary to the maintaining of steady motion when the body is projected in the direction of its axis. The equations for $\dot{x}$, $\dot{y}$, $\dot{z}$ now give after integration

56. $$\begin{aligned}x &= \frac{L}{\eta}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) \sin \theta' \cos \theta' \sin \eta t, \\ y &= \frac{L}{\eta}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) \sin \theta' \cos \theta' \cos \eta t, \\ z &= L\left(\frac{\cos^2 \theta'}{a_{33}} + \frac{\sin^2 \theta'}{a_{11}}\right) t,\end{aligned}$$

or

57. $$\begin{aligned}\frac{x}{y} &= \tan \eta t \\ z &= L \cos^2 \theta' \left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right) t + \frac{Lt}{a_{11}},\end{aligned}$$

the equations of a helix whose projection on the plane of xy is the circle

$$x^2 + y^2 = \frac{L^2}{\gamma^2}\left(\frac{1}{a_{33}} - \frac{1}{a_{11}}\right)^2 \sin^2\theta' \cos^2\theta'. \tag{58}$$

This is in accordance with what was proved in the paper already referred to, viz: that a body moving in a fluid has an infinite number of possible steady motions, each of which consists of a twist about a certain screw.

The problem of the motion of a solid of revolution in a fluid has been very fully solved by Kirchhoff in Borchardt's Journal, Bd. 71, and by others since that time. I append a list of the more important articles on this subject:

Clebsch: Die Bewegung eines Ellipsoids in einer Flüssigkeit. Crelle's Journal, Bd. 52 and 53.

Kirchhoff: Ueber die Bewegung eines Rotations Körpers in einer Flüssigkeit. Borchardt's Journal, Bd. 71.

Ferrers: The motion of an infinite mass of water about a moving Ellipsoid. Quart. Jour., No. 52. 1875.

Köpcke: Zur Discussion der Bewegung eines Rotations Körpers in einer Flüssigkeit. Math. Annalen., Vol. 12, p. 387.

Weber: Anwendung der Thetafunctionen zweier Veränderlichen auf die Theorie der Bewegung eines festen Körpers in einer Flüssigkeit. Math. Annalen., Vol. 14, p. 173.

The motion of a sphere in a fluid was treated by Dirichlet, and was the first article of note which treated of the motion in a fluid of a body of given form.

WASHINGTON, *July 5, 1879.*